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A DYNAMIC CONTACT PROBLEM FOR A BILAYERED HALF-SPACE WITH A CAVITY*

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A dynamic contact problem for a bilayered elastic half-space with a deep cylindrical cavity. The superposition principle, that enables the boundary value problem to be reduced to a system of integral equations, solved by successive approximations in the case of intense cavity depth in the underlying half-space, is used to construct the solution. The solution of an integral equation of the first kind, whose operator corresponds to the solution of a contact problem for a bilayered half-space without a cavity is constructed at each stage of application of the method. The free term of the integral equation is a power series in a small parameter determining the ratio between the wavelength and the magnitude of the depth of the centre of the cavity. To realize the algorithm the solution of the singular integral equation is constructed by using approximate factorization of functions. The degree of influence of the cavity on the distribution law and the magnitude of the contact stresses under the stamp is investigated.

We consider the problem of exciting antiplane steady harmonic vibrations in an elastic bilayered half-space with a horizontal cylindrical cavity of relatively small radius. It is assumed that the cavity is completely in the underlying half-space. Let us formulate the appropriate boundary value problem. Let an elastic medium occupy a domain

$$x \geqslant -H \cup R = \sqrt{(\varepsilon^{-1}-x)^2+y^2} \geqslant 1, \quad \varepsilon = a/h$$

in a dimensionless Cartesian xyz system of coordinates, characterized by values ρ_1, λ_1, μ_1 of the density and Lamé coefficients for $x \geqslant 0$ and ρ, λ, μ for $-H \leqslant x \leqslant 0$. We obtain the dimensional coordinates by multiplying the dimensionless coordinates by the cavity radius a, while h is the magnitude of the depth of the cavity centre in the half-space.

The following conditions are given on the boundary of the elastic medium:

$$\begin{gathered} x=-H, \quad u_z = f(y)e^{-i\omega t}, \quad y \in I; \ \tau_{xz}=0, \ y \,\overline{\in}\, I, \quad I=[c-b, \ c+b] \\ R=1, \quad \tau_{Rz}=\tau(\varphi)e^{-i\omega t} \end{gathered} \tag{1}$$

The radiation conditions for the elastic wave energy are given at infinity. The motion of the medium is described by the Lamé equations /1/.

To a substantial degree, the solution of contact problems of elasticity theory rely on an investigation of appropriate problems with homogeneous boundary conditions. The solution of the homogeneous problem for a bilayered space with a cavity relatively deep in a half-space is constructed by using the superposition principle /2/ and enables us to analyse the corresponding boundary value problem with mixed boundary conditions on a plane surface.

Let

$$\tau_{xz} = \begin{cases} t(y)e^{-i\omega t}, & y \in I \\ 0, & y \,\overline{\in}\, I \end{cases} \tag{2}$$

denote the unknown contact stresses.

The problem with homogeneous boundary conditions on plane and cylindrical boundaries reduces to a system of two integral equations in terms of whose solution the displacement field in the whole elastic domain is described

$$\bar{X}(\alpha) - \sum_{m=-\infty}^{\infty} Y_m L_m(\alpha) = \bar{T}(\alpha), \; Y(\varphi) - \frac{1}{2\pi}\int_\Gamma \bar{X}(\alpha)K(\varphi,\alpha)\,d\alpha = \tau(\varphi) \tag{3}$$

$$\bar{X}(\alpha) = \int_{-\infty}^{\infty} X(y)e^{i\alpha y}\,dy, \qquad Y_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\varphi)e^{-im\varphi}\,d\varphi$$

$$K(\varphi,\alpha) = [\cos\varphi + i\alpha\sigma_1^{-1}\sin\varphi]\exp[-\sigma_1(\varepsilon^{-1}-\cos\varphi) + i\alpha\sin\varphi]$$

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$$L_m(\alpha) = \frac{\sigma_1 \operatorname{ch} \sigma H}{\Delta_m \delta(\alpha)} K_m(\alpha), \quad \sigma = \sqrt{\alpha^2 - \theta^2}, \quad \sigma_1 = \sqrt{\alpha^2 - \theta_1^2}$$

$$K_m(\alpha) = \bar{K}_{1m}(\alpha) + \int_{-\infty}^{\infty} \gamma\sigma H_m^{(1)}\left(\frac{\theta_1}{\varepsilon}\sqrt{1+\varepsilon^2 y^2}\right) \operatorname{th} \sigma H \times$$
$$\exp\left[-im \operatorname{arctg}(\varepsilon y) + i\alpha y\right] dy$$

$$\bar{K}_{1m}(\alpha) = \int_{-\infty}^{\infty} \left\{\left[\frac{m\varepsilon}{\theta_1\sqrt{1+\varepsilon^2 y^2}} H_m^{(1)}\left(\frac{\theta_1}{\varepsilon}\sqrt{1+\varepsilon^2 y^2}\right) - \right.\right.$$
$$\left.\theta_1 H_{m+1}^{(1)}\left(\frac{\theta_1}{\varepsilon}\sqrt{1+\varepsilon^2 y^2}\right)\right] \frac{1}{\sqrt{1+\varepsilon^2 y^2}} +$$
$$\left.\frac{im\varepsilon^2 y}{1+\varepsilon^2 y^2} H_m^{(1)}\left(\frac{\theta_1}{\varepsilon}\sqrt{1+\varepsilon^2 y^2}\right)\right\} - \exp\left[-im \operatorname{arctg}(\varepsilon y) + i\alpha y\right] dy$$

$$\bar{T}(\alpha) = \frac{\sigma_1 \bar{t}(\alpha)}{\delta(\alpha) \operatorname{ch} \sigma H}, \quad \theta = \omega a \sqrt{\frac{\rho}{\mu}}, \quad \theta_1 = \omega a \sqrt{\frac{\rho_1}{\mu_1}}, \quad \gamma = \frac{\mu}{\mu_1}$$

$$\Delta_m = m H_m^{(1)}(\theta_1) - \theta_1 H_{m+1}^{(1)}(\theta_1), \quad \delta(\alpha) = \sigma_1 \operatorname{ch} \sigma H + \gamma\sigma \operatorname{sh} \sigma H$$

The displacement field in an elastic medium is described in terms of the solution of system (3) by relationships of the form

$$u_z = u_z^{(1)} + u_z^{(2)}, \quad x \geqslant 0 \cup R \geqslant 1 \tag{4}$$
$$u_z = u_z^{(3)}, \quad -H \leqslant x \leqslant 0$$
$$u_z^{(1)}(x, y) = -\frac{a}{2\pi\mu_1} \int_\Gamma \bar{X}(\alpha) \sigma_1^{-1} \exp\left[-\sigma_1 x - i\alpha y\right] d\alpha$$
$$u_z^{(2)}(R, \varphi) = \frac{a}{\mu_1} \sum_{m=-\infty}^{\infty} Y_m \Delta_m^{-1} H_m^{(1)}(\theta_1 R) \exp(im\varphi)$$
$$u_z^{(3)}(x, y) = -\frac{a}{2\pi\mu_1} \int_\Gamma \frac{e^{-i\alpha y}}{\sigma \operatorname{sh} \sigma H} \left[\bar{t}(\alpha) \operatorname{ch} \sigma x - \bar{Z}(\alpha) \operatorname{ch}(\sigma[x+H])\right] d\alpha$$
$$\bar{Z}(\alpha) = \frac{\sigma_1 \bar{t}(\alpha)}{\delta(\alpha)} - \frac{\gamma\sigma \operatorname{sh} \sigma H}{\delta(\alpha)} \sum_{m=-\infty}^{\infty} Y_m \left\{\bar{K}_{1m}(\alpha) - \right.$$
$$\left.\int_{-\infty}^{\infty} \Delta_m^{-1} \sigma_1 H_m^{(1)}\left(\frac{\theta_1}{\varepsilon}\sqrt{1+\varepsilon^2 y^2}\right) \exp\left[i\left(m \operatorname{arctg}(-\varepsilon y) + \alpha y\right)\right] dy\right\}$$

The shape of the contour Γ is determined by application of the limit absorption principle and is described in /2, 3/.

Returning to the original boundary value problem with mixed boundary conditions, we have an unknown function $t(y)$ in system (3) that characterises the contact stress distribution law, i.e., a system of two integral equations with three unknowns $t(y)$, $X(y)$, and $Y(\varphi)$ is obtained. We use the first boundary condition of (1) to close the system. Consequently, taking (4) into account, we obtain a third integral equation that closes the system (3),

$$u_z^{(3)} = u_z^{(1)} + u_z^{(2)}, \quad x = 0. \tag{5}$$

Solving the first equation of (3) for $\bar{X}(\alpha)$ and substituting the value obtained into (5), we write the last equation in the form

$$\int_\Gamma \bar{t}(\alpha) M(\alpha) e^{-i\alpha y} d\alpha = \frac{2\pi}{a} f(y) - \sum_{m=-\infty}^{\infty} Y_m \int_\Gamma M_m(\alpha) e^{-i\alpha y} d\alpha \tag{6}$$

$$M(\alpha) = -[\gamma\sigma \operatorname{ch} \sigma H + \sigma_1 \operatorname{sh} \sigma H]/[\mu\sigma\delta(\alpha)] \tag{7}$$
$$M_m(\alpha) = -\frac{\gamma}{\mu\delta(\alpha)} \left[K_m(\alpha) - \int_{-\infty}^{\infty} \frac{\sigma_1}{\Delta_m} H_m^{(1)}\left(\frac{\theta_1}{\varepsilon}\sqrt{1+\varepsilon^2\eta^2}\right) \times\right.$$
$$\left.\exp\left\{-i\left[m \operatorname{arctg}(\varepsilon\eta) - \alpha\eta\right]\right\} d\eta\right], \quad y \in I$$

We will examine the properties of system (3) and (6) for $\varepsilon \ll 1, \varepsilon/\theta \ll 1$. In this case the operator of system (3) will be completely continuous and small for sufficiently small ε. The left side (6) contains an integral operator that exactly corresponds to the operator of the integral equation of the contact problem for an elastic bilayered half-space without a cavity /4/. The operators on the right-hand side of (6) are completely continuous and small within the framework of the constraints introduced.

The properties mentioned for system (3) and (6) enable us to use successive approximations for its solution. To a first approximation it follows from (3) that

$$\bar{X}(\alpha) = \bar{t}(\alpha) - O(\sqrt{\varepsilon}), \quad Y(\varphi) = \tau(\varphi) + O(\sqrt{\varepsilon})$$

Substituting the values obtained into the right-hand side of (6) and retaining the principal terms of the expansion, we obtain an integral equation of the first kind

$$\int_{\Gamma} \bar{t}(\alpha) M_{.}(\alpha) e^{-i\alpha y} d\alpha = \frac{2\pi}{a} f(y), \quad y \in I \tag{8}$$

to determine $t(y)$ to a first approximation, and to which the solution of the contact problem for a bilayered half-space without a cavity reduces /4/.

Effective methods exist for constructing the solution of equations of the kind mentioned /3, 5, 6/. We use the method of /4/ to construct the solution. From the results of computations we approximate $t_0(y)$ to a given degree of accuracy to a first approximation by an expression of the form

$$t_0(y) = t_0 \frac{p(y_*)}{\sqrt{b^2 - y_*^2}}; \quad p(y_*) = \sum_{j=0}^{N} p_j y_*^j, \quad y_* = y - c, \quad t_0 = \text{const}$$

Then

$$\bar{t}_0(\alpha) = \pi t_0 e^{i\alpha c} \sum_{k=0}^{N} p_k (-1)^k \frac{d^k}{d\alpha^k} [J_0(\alpha b)]$$

Substituting the value of $\bar{t}_0(\alpha)$ obtained into the right-hand side of (3), we have to a second approximation

$$Y_m \cong \tau_m + \sqrt{\frac{\varepsilon\theta_1}{2\pi}} \left\{ \bar{T}(\alpha_0) + \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} L_k(\alpha_0) \tau_k \right\} e^{-i\pi/4} \times$$

$$\left[\left(1 + \frac{\alpha_0}{\sqrt{\theta_1^2 - \alpha_0^2}}\right) J_{m-1}(\theta) - \left(1 - \frac{\alpha_0}{\sqrt{\theta_1^2 - \alpha_0^2}}\right) J_{m+1}(\theta) \right] \frac{(-i)^{m-1}}{2} + O(\varepsilon^{3/2})$$

$$\bar{X}(\alpha) = \bar{T}(\alpha) + \sum_{m=-\infty}^{\infty} Y_m L_m(\alpha), \quad \alpha_0 = -\frac{\theta_1 \varepsilon y}{\sqrt{1 + \varepsilon^2 y^2}}$$

Substituting the last expression for $\bar{t}_0(\alpha)$ into the right-hand side of (6), we arrive at the necessity for the solution of (8) with changed right-hand side, equal to

$$f_1(y) = \frac{2\pi}{a}(f - y) \sum_{m=-\infty}^{\infty} Y_m M_m^*(y)$$

$$M_m^*(y) = -\frac{2i\sqrt{2\varepsilon\theta_1/\pi}}{\mu_1 \Delta_m \delta(\alpha_0)} (1 + \varepsilon^2 y^2)^{-1/4} \exp\left\{ i \left[m \operatorname{arctg}(-\varepsilon y) + \right.\right.$$

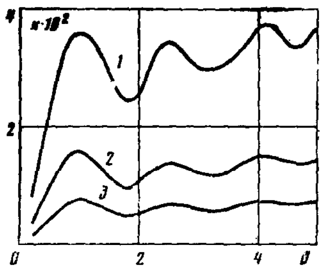
$$\left.\left. \theta_1 \varepsilon^{-1} \sqrt{1 + \varepsilon^2 y^2} - \left(\frac{m}{2} + \frac{1}{4}\right)\pi \right] \right\} + O(\varepsilon^{3/2})$$

$$L_m(\alpha) = -2i \frac{-\sigma_1 \operatorname{ch} \sigma H + \gamma\sigma \operatorname{sh} \sigma H}{\Delta_m \delta(\alpha)} \times$$

$$\exp\left\{ i \left[\varepsilon^{-1} \sqrt{\theta_1^2 - \alpha^2} - \frac{m\pi}{2} + m \operatorname{arctg} \frac{\alpha}{\sqrt{\theta_1^2 - \alpha^2}} \right] \right\} + O(\varepsilon)$$

Therefore, application of successive approximations results in the need to construct the solution of a singular integral equation of the first kind at each stage of the process, to which the appropriate boundary value problem for a bilayered elastic half-space without a cavity reduces, whose operator part does not change from approximation to approximation while only the free term does. The correction terms have an order of smallness that increases from approximation to approximation and are determined, in practice, from the solution of the homogeneous problem for the approximate domain.

It should be noted that for $H = 0$ all the expressions presented above degenerate into the appropriate expressions for the problem for a homogeneous half-space. The latter enables computations to be carried out for the correction components on the right-hand side of (8) according to a single program for both problems. The essential difference in the properties of the integrand in the kernel of this equation governs application of a single approach to the construction of the solution (for instance 3). However, taking account of specific singularities of the kernel that determine the fundamental physical properties of the solution (the presence of singularities on the real axis, the domains of complexity of the integrand, etc.) is required for its practical realization, expecially at the stage of constructing the approximate factorization.



The scheme elucidated for solving the boundary value problem was realized on an electronic computer by applied programs that enable the degree of influence of cavities of relatively small radius on the contact stress distribution under a stamp to be analysed. The relative error in the solution k due to the presence of a cavity is given in the figure as a function of the dimensionless vibration frequency for cavity depths of $h = 20, 50, 100$ (curves *1*, *2*, *3*, respectively).

It should be noted that for a sufficient distance of the cavity from the interface between the layer and the half-space ($\varepsilon/\theta_1 < 0.1$) and a relatively small contact zone ($\varepsilon^{-1}\theta_1 \gg h\theta$) the correction component of the right-hand side of (8) does not lead, in practice, to distortion of the law of free term variation along the coordinate y. Only the magnitude of the factor for $f(y)$ changes, which governs the perturbation of the contact stress magnitude for an insignificant distortion of their distribution law in the contact domain from approximation to approximation.

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